

JACOB'S LADDERS AND THE MULTIPLICATIVE ASYMPTOTIC FORMULA FOR SHORT AND MICROSCOPIC PARTS OF THE HARDY-LITTLEWOOD INTEGRAL

JAN MOSER

ABSTRACT. The elementary geometric properties of Jacob's ladders lead to a class of new asymptotic formulae for short and microscopic parts of the Hardy-Littlewood integral. This class of asymptotic formulae cannot be obtained by methods of Balasubramanian, Heath-Brown and Ivic.

Dedicated to 110 anniversary of E.C. Titchmarsh.

1. FORMULATION OF THE THEOREM

In the papers [4] and 5 I obtained the following additive formula

$$(1.1) \quad \int_T^{T+U} Z^2(t) dt = U \ln \left(\frac{\varphi(T)}{2} e^{-a} \right) \tan[\alpha(T, U)] + \mathcal{O} \left(\frac{1}{T^{1/3-4\epsilon}} \right),$$

$$U \in (0, U_0], \quad U_0 = T^{1/3+2\epsilon}, \quad a = \ln 2\pi - 1 - c,$$

that holds true for short parts of the Hardy-Littlewood integral. In the present work I prove a multiplicative formula, which is asymptotic at $T \rightarrow \infty$ also if $U \rightarrow 0$. Namely, the following theorem takes place

Theorem.

$$(1.2) \quad \int_T^{T+U} Z^2(t) dt = U \ln T \tan[\alpha(T, U)] \left\{ 1 + \mathcal{O} \left(\frac{\ln \ln T}{\ln T} \right) \right\}, \quad U \in \left(0, \frac{T}{\ln T} \right],$$

for $\mu[\varphi] = 7\varphi \ln \varphi$

The main idea of the proof of the Theorem is to use the formula

$$(1.3) \quad Z^2(t) = \Phi'[\varphi(T)] \frac{d\varphi(T)}{dT}, \quad \Phi' = \Phi'_\varphi,$$

where

$$(1.4) \quad \Phi'[\varphi] = \frac{2}{\varphi^2} \int_0^{\mu[\varphi]} t e^{-\frac{2}{\varphi}t} Z^2(t) dt + Z^2\{\mu[\varphi]\} e^{-\frac{2}{\varphi}\mu[\varphi]} \frac{d\mu[\varphi]}{d\varphi},$$

(see [4], (3.5), (3.9)).

Remark 1. In the proof of the formula (1.2) we shall get also the multiplicative formula

$$(1.5) \quad Z^2(T) = \frac{1}{2} \ln T \frac{d\varphi(T)}{dT} \left\{ 1 + \mathcal{O} \left(\frac{\ln \ln T}{\ln T} \right) \right\}, \quad T \rightarrow \infty.$$

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Remark 2. Our new method leads to asymptotic formulae (see, e.g. (2.3), (2.4)) for short and microscopic parts of the Hardy-Littlewood integral. These asymptotic formulae cannot be derived within complicated methods of Balasubramanian, Heath-Brown and Ivic (see [1] and estimates in [3], pp. 178 and 191).

2. CONSEQUENCES OF THE THEOREM

2.1. First of all, we will show a canonical equivalence that follows from (1.2). Let us remind that we call the chord binding the points

$$(2.1) \quad \left[T, \frac{1}{2}\varphi(T) \right], \left[T + U_0, \frac{1}{2}\varphi(T + U_0) \right], \tan[\alpha(T, U_0)] = 1 + \mathcal{O}\left(\frac{1}{\ln T}\right)$$

of the Jacob's ladder $y = \frac{1}{2}\varphi(T)$ the *fundamental chord* (see [4]).

Definition. The chord binding the points

$$\left[N, \frac{1}{2}\varphi(N) \right], \left[M, \frac{1}{2}\varphi(M) \right], [N, M] \subset [T, T + U_0],$$

for which the property

$$\tan[\alpha(N, M - N)] = 1 + o(1), \quad T \rightarrow \infty$$

is fulfilled, is called *the almost parallel chord* to the fundamental chord. This property we will denote by the symbol \parallel .

Corollary 1. Let $[N, M] \subset [T, T + U_0]$. Then

$$(2.2) \quad \frac{1}{M - N} \int_N^M Z^2(t) dt \sim \ln T \Leftrightarrow \parallel.$$

Remark 3. We see that the analytic property

$$\frac{1}{M - N} \int_N^M Z^2(t) dt \sim \ln T$$

is equivalent to the geometric property \parallel of Jacob's ladder $y = \frac{1}{2}\varphi(T)$.

2.2. Next, for example, similarly to the case of the paper [5], Cor. 1, we obtain from our Theorem

Corollary 2. There is continuum of intervals $[N, M] \subset [T, T + U_0]$ for which the following asymptotic formula

$$(2.3) \quad \int_N^M Z^2(t) dt = (M - N) \ln T \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln T}{\ln T}\right) \right\}.$$

holds true.

Remark 4. Especially, there is continuum of intervals $[N, M] : 0 < M - N < 1$ for which the asymptotic formula (2.3) holds true (this follows from the elementary mean value theorem of differentiation).

And similarly to [5], Cor. 3, part A, we obtain from our Theorem

Corollary 3. For every sufficiently big zero $T = \gamma$ of the function $\zeta(\frac{1}{2} + iT)$ there is continuum of intervals $[\gamma, U(\gamma, \alpha)]$ such that the following is true

$$(2.4) \quad \int_{\gamma}^{\gamma + U(\gamma, \alpha)} Z^2(t) dt = U \ln \gamma \tan \alpha \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln \gamma}{\ln \gamma}\right) \right\},$$

where $\tan \alpha \in [\eta, 1 - \eta]$, and α is the angle of the rotating chord binding the points

$$\left[\gamma, \frac{1}{2}\varphi(\gamma) \right], \left[\gamma + U, \frac{1}{2}\varphi(\gamma + U) \right],$$

and $0 < \eta$ is an arbitrarily small number.

Remark 5. For example, in the case $\alpha = \pi/6$ we have, as a special case of eq. (2.4),

$$\int_{\gamma}^{\gamma+U(\gamma, \pi/6)} Z^2(t) dt \sim \frac{1}{\sqrt{3}} U \ln \gamma.$$

Remark 6. It is obvious that

$$U(\gamma, \alpha) < T^{1/3+2\epsilon}.$$

Moreover, the following is also true

$$U(\gamma, \alpha) < T^{\omega}, \quad \omega \in \left[\frac{1}{4} + \epsilon, \frac{1}{3} + 2\epsilon \right),$$

(compare the Good's Ω -Theorem [2]), where ω can attain every value for which the formula

$$\int_0^T Z^2(t) dt = T \ln T + (2c - 1 - \ln 2\pi)T + \mathcal{O}(T^{\omega})$$

will be proved.

3. AN ESTIMATE FOR $\Phi''_{y^2}[\varphi]$

The following lemma is true

Lemma 1. *If $\mu[\varphi] = 7\varphi \ln \varphi$ then*

$$(3.1) \quad \Phi''_{y^2}[\varphi] = \mathcal{O}\left(\frac{1}{\varphi} \ln \varphi \ln \ln \varphi\right).$$

Proof. Since $\mu(y) = 7y \ln y$ we have

$$\mu(y) \rightarrow y = \varphi_{\mu}(T) = \varphi(T) = \varphi,$$

see [4] and (1.4),

$$(3.2) \quad \Phi''_{y^2}[\varphi] = \frac{4}{\varphi^3} \int_0^{\mu[\varphi]} t \left(\frac{t}{\varphi} - 1 \right) e^{-\frac{2}{\varphi}t} Z^2(t) dt + Q[\varphi],$$

$$(3.3) \quad Q[\varphi] = e^{-\frac{2}{\varphi}\mu[\varphi]} \left\{ \frac{2}{\varphi^2} Z^2\{\mu[\varphi]\} \mu[\varphi] \frac{d\mu[\varphi]}{d\varphi} + \frac{2}{\varphi^2} Z^2\{\mu[\varphi]\} \frac{d\mu[\varphi]}{d\varphi} - \frac{2}{\varphi} Z^2\{\mu[\varphi]\} \left(\frac{d\mu[\varphi]}{d\varphi} \right)^2 + 2Z\{\mu[\varphi]\} Z'_{\mu}\{\mu[\varphi]\} \left(\frac{d\mu[\varphi]}{d\varphi} \right)^2 + Z^2\{\mu[\varphi]\} \frac{d^2\mu[\varphi]}{d\varphi^2} \right\}.$$

Let

$$g(t) = t \left(\frac{t}{\varphi} - 1 \right) e^{-\frac{2}{\varphi}t}, \quad t \in [0, \mu[\varphi]].$$

We apply the following elementary facts

$$\begin{aligned}
 (3.4) \quad & g(0) = g(\varphi) = 0, \quad g' \left[\left(1 - \frac{1}{\sqrt{2}}\right) \varphi \right] = g' \left[\left(1 + \frac{1}{\sqrt{2}}\right) \varphi \right] = 0, \\
 & \min\{g(t)\} = -\frac{1}{\sqrt{2}} \left(1 - \frac{1}{\sqrt{2}}\right) e^{-2+\sqrt{2}} \varphi \\
 & \max\{g(t)\} = \frac{1}{\sqrt{2}} \left(1 + \frac{1}{\sqrt{2}}\right) e^{-2-\sqrt{2}} \varphi \\
 & g(t) \leq g(\varphi \ln \ln \varphi) < \varphi \left(\frac{\ln \ln \varphi}{\ln \varphi} \right)^2, \quad t \in [\varphi \ln \ln \varphi, 7\varphi \ln \varphi],
 \end{aligned}$$

and the Hardy-Littlewood formula (1918)

$$(3.5) \quad \int_0^T Z^2(t) dt \sim T \ln T, \quad T \rightarrow \infty.$$

First of all we have

$$\begin{aligned}
 (3.6) \quad & \frac{4}{\varphi^3} \int_0^{\varphi \ln \ln \varphi} Z^2(t) dt = \mathcal{O} \left(\frac{1}{\varphi^2} \int_0^{\varphi \ln \ln \varphi} Z^2(t) dt \right) = \mathcal{O} \left(\frac{1}{\varphi} \ln \varphi \ln \ln \varphi \right), \\
 & \frac{4}{\varphi^3} \int_{\varphi \ln \ln \varphi}^{7\varphi \ln \varphi} Z^2(t) dt = \mathcal{O} \left\{ \frac{1}{\varphi^3} \varphi \left(\frac{\ln \ln \varphi}{\ln \varphi} \right)^2 \varphi \ln^2 \varphi \right\} = \mathcal{O} \left\{ \frac{1}{\varphi} (\ln \ln \varphi)^2 \right\}
 \end{aligned}$$

by (3.4), (3.5). Next we have (see (3.3))

$$(3.7) \quad Q[\varphi] = \mathcal{O}(\varphi^{-13}) \rightarrow 0, \quad T \rightarrow \infty.$$

Finally, we obtain (3.1) from (3.2) by (3.6) and (3.7). □

Remark 7. It is quite evident that our Lemma (i.e. also our Theorem) is true for continual class of functions

$$\mu[\varphi] = 7\varphi^{\omega_1} \ln^{\omega_2} \varphi, \quad \omega_1, \omega_2 \geq 1.$$

4. PROOF OF THE THEOREM

By (1.3) we have

$$\int_T^{T+U} Z^2(t) dt = \Phi'_y[\varphi(t_1)] \int_T^{T+U} d\varphi = \Phi'_y[\varphi(t_1)] \{\varphi(T+U) - \varphi(T)\},$$

i.e.

$$\begin{aligned}
 (4.1) \quad & \int_T^{T+U} Z^2(t) dt = 2U \Phi'_y[\varphi(t_1)] \tan[\alpha(T, U)], \quad t_1 = t_1(U) \in (T, T+U), \\
 & \tan[\alpha(T, U)] = \frac{1}{2} \frac{\varphi(T+U) - \varphi(T)}{U}.
 \end{aligned}$$

Next we have

$$(4.2) \quad \int_T^{T+U_0} Z^2(t) dt = 2U_0 \Phi'_y[\varphi(t_2)] \left\{ 1 + \mathcal{O} \left(\frac{1}{\ln T} \right) \right\}, \quad t_2 = t_2(U_0) \in (T, T+U_0),$$

by (2.1), (4.1). Let us remind that $\varphi(T)/2 \sim T$ by the formula

$$(4.3) \quad \pi(T) \sim \frac{1}{1-c} \left\{ T - \frac{\varphi(T)}{2} \right\}, \quad T \rightarrow \infty,$$

(see [4], (6.2)). Hence by comparison of the formulae (1.1) $U = U_0$ (see (2.1)) and (4.2) we obtain

$$(4.4) \quad \Phi'_y[\varphi(t_2)] = \frac{1}{2} \ln T + \mathcal{O}(1).$$

We have by (4.3)

$$\varphi(t_1) - \varphi(t_2) = 2(t_1 - t_2) + \mathcal{O}\left(\frac{T}{\ln T}\right) = \mathcal{O}\left(\frac{T}{\ln T}\right), \quad U \in \left(0, \frac{T}{\ln T}\right],$$

and subsequently

$$(4.5) \quad \Phi'_y[\varphi(t_1)] - \Phi'_y[\varphi(t_2)] = \mathcal{O}\left\{|\Phi''_{y^2}(T)| \cdot |\varphi(t_1) - \varphi(t_2)|\right\} = \mathcal{O}(\ln \ln T),$$

and therefore

$$(4.6) \quad \Phi'_y[\varphi(t_1)] = \frac{1}{2} \ln T + \mathcal{O}(\ln \ln T),$$

by (4.4), (4.5). Finally, (1.2) follows (4.1), (4.6).

Similarly to (4.5) we have

$$(4.7) \quad \Phi'_y[\varphi(t_1)] - \Phi'_y[\varphi(T)] = \mathcal{O}(\ln \ln T).$$

Then we obtain (1.5) by (4.6), (4.7).

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DEPARTMENT OF MATHEMATICAL ANALYSIS AND NUMERICAL MATHEMATICS, COMENIUS UNIVERSITY, MLYNSKA DOLINA M105, 842 48 BRATISLAVA, SLOVAKIA
E-mail address: jan.moser@fmph.uniba.sk